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Exact Solution of a Triangular Ising Model in a Nonzero Magnetic Field[1]

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A closed-form expression is obtained for the free energy per site of the Ising model on the triangular lattice in a nonzero magnetic field and with two- and three-site interactions. The solution is valid along a trajectory in the parameter space, and is derived using a method of exact decimation. A criterion determining the validity of the decimation method is also established.

KEY WORDS: Ising model; exact solution; nonzero magnetic field.

1. INTRODUCTION

One outstanding unsolved problem in statistical mechanics is the closed-form computation of the free energy of the two-dimensional Ising model in a nonzero magnetic field. In 1976 Verhagen[(1)] considered one particular triangular Ising model, and obtained its solution along a certain trajectory in the parameter space. This solution, which was obtained through the consideration of a stochastic crystal growth model, has since been extended by Ruján[(2)] to the fully isotropic antiferromagnetic model with nearest-neighbor interactions. Quite recently, it has been further recognized that the nonzero field triangular Ising model is related to a number of other important two-dimensional lattice-statistical problems. The nearest-neighbor model is shown to relate to the problem of directed lattice animals,[(3)] and the Ising model with two- and three-spin interactions is equivalent to cellular automata and directed percolation.[(4)] Therefore, it is not without interest to seek for further solutions of the triangular Ising system. In this connection it should be pointed out that the hard-hexagon problem solved by Baxter[(5)] corresponds to an infinite-field, infinite-

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interaction limit of the antiferromagnetic triangular Ising model. Enting[6] has also formulated the general triangular Ising model as a crystal growth model, and considered the soluble cases for zero field and isotropic interactions. It appears that the corresponding solution for the general (anisotropic) triangular model has not been previously given; it is also desirable to have alternate, and hopefully simpler, derivations of the solution without the recourse of the intermediate step of a crystal growth model.[2]

It is the purpose of this paper to provide such a solution.[8] We consider the general triangular Ising model with two- and three-spin interactions in a nonzero magnetic field as shown in Fig. 1, and use a simple decimation procedure to derive its free energy along a trajectory in the parameter space. For zero three-spin interactions our trajectory reduces to that considered by Ruján,[2] but our procedure permits a simple and direct evaluation of the free energy. We also obtain a criterion which determines the validity of the decimation procedure.

Consider an Ising model on a triangular lattice, shown in Fig. 1, of N sites with the Hamiltonian

$$\mathscr{H} = -\sum_{\Delta} (J_1\sigma_2\sigma_3 + J_2\sigma_3\sigma_1 + J_3\sigma_1\sigma_2 + J\sigma_1\sigma_2\sigma_3) - H\sum_i \sigma_i \qquad (1)$$

where H is the external magnetic field, and the summation $\sum_{\Delta}$ is taken over all up-pointing triangular faces, shaded in Fig. 1, of the lattice. Our goal is to compute the partition function per site

$$\kappa = \lim_{N\to\infty} Z^{1/N} \qquad (2)$$

where Z is the partition function defined by (1).

2. THE SOLUTION

Orient the lattice as shown in Fig. 1 where a periodic boundary condition is imposed in the horizontal direction only. We modify the Hamiltonian (1) by changing the external field applied to the spins located on the upper boundary to a new value H'. Then each term in $e^{-mL'}Z$ is a positive Boltzmann weight multiplied to a factor which is a power of $e^{2L'}$,

[2] After the completion of this work we received a preprint from M. T. Jaekel and J. M. Maillard[7] who formulated the decimation method adopted here for general spin models, and obtained (18) and (19) for the nearest-neighbor triangular Ising model as a special application. They did not, however, discuss the effect of the boundary field and the associated validity conditions.

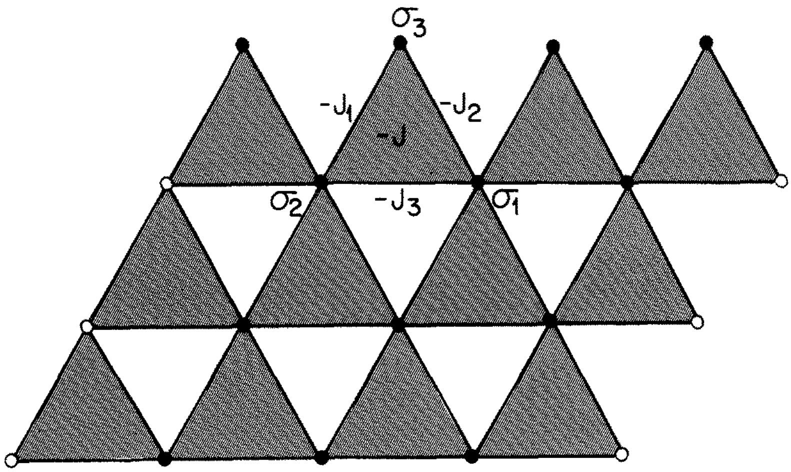

Fig. 1. Triangular lattice with periodic condition in the horizontal direction. The open circles in the same row indicate the same site.

where $L' = H'/kT$ and m is the number of columns of the lattice. It follows that the modification of the boundary field will not change the bulk free energy $\ln \kappa$ provided that[3,4]

$$e^{2L'} > 0 \qquad \text{or} \qquad \cosh 2L' > 1 \tag{3}$$

We take $m = \text{even}$ so Z is always positive.

The particular boundary condition we chose permits us to carry out the partition sums over the spins located in the upper boundary row. In fact, we carry out the spin sums over in the first row of spins in Fig. 1, and require

$$\sum_{\sigma_3} \exp(L'\sigma_3 + K\sigma_1\sigma_2\sigma_3 + K_1\sigma_2\sigma_3 + K_2\sigma_3\sigma_1 + K_3\sigma_1\sigma_2)$$
$$= F\exp(L_1^*\sigma_1 + L_2^*\sigma_2) \tag{4}$$

[3] If $e^{2L'}$ is negative (or complex), then terms in the partition function may cancel and the bulk free energy (2) will be dependent on the boundary field. This is borne out by the $J = 0$ solution in Sec. 3.

[4] Strictly speaking, modification of the boundary field requires that the LHS of (4) multiplied by $e^{L'}$ is positive. Then (13) leads to generally complex field L if (3) does not hold.

where $K_i = J_i/kT$, $K = J/kT$. Explicitly, (4) is written as

$$\begin{aligned}
2e^{K_3}\cosh(L'+K+K_1+K_2) &= Fe^{L_1^*+L_2^*}\\
2e^{K_3}\cosh(L'+K-K_1-K_2) &= Fe^{-L_1^*-L_2^*}\\
2e^{-K_3}\cosh(L'-K+K_1-K_2) &= Fe^{-L_1^*+L_2^*}\\
2e^{-K_3}\cosh(L'-K-K_1+K_2) &= Fe^{L_1^*-L_2^*}
\end{aligned} \tag{5}$$

Using the four equations given by (5) we can uniquely determine the four unknowns L', F, L_1^*, and L_2^*. This leads to

$$F^2 = -2(\sinh 2K_1 \sinh 2K_2 + \sinh 2K \sinh 2L')/\sinh 2K_3 \tag{6}$$

$$e^{2(L_1^*+L_2^*)} = \cosh(L'+K+K_1+K_2)/\cosh(L'+K-K_1-K_2) \tag{7}$$

where L' is to be determined from

$$Ae^{2L'} + B + Ce^{-2L'} = 0 \tag{8}$$

with

$$\begin{gathered}
A = \sinh 2(K_3+K), \qquad C = \sinh 2(K_3-K)\\
B = e^{2K_3}\cosh 2(K_1+K_2) - e^{-2K_3}\cosh 2(K_1-K_2)
\end{gathered} \tag{9}$$

Now the quadratic equation $ax^2+bx+c=0$ has a positive solution only if $ac<0$, or $ac>0$, $ab<0$, and $b^2 \geqslant 4ac$. It then follows from (3) and (8) that our procedure is valid for

$$|J| > |J_3| \tag{10}$$

and for

$$|J| < |J_3| \tag{11}$$

provided that

$$B^2 \geqslant 4AC, \qquad K_3 B < 0 \tag{12}$$

The particular form of the Boltzmann weight on right-hand side of (4) indicates that the partition sums have, in effect, decimated the first row of shaded triangles (cf. Fig. 1). The remaining lattice is an exact copy of the original one except that it has one less row. The new boundary spins now have fields $L + L_1^* + L_2^*$, where $L = H/kT$. If L further satisfies

$$L + L_1^* + L_2^* = L' \tag{13}$$

then we can repeat the decimation process by summing over the new boundary spins. Continuing in this fashion, we eventually decimate all spins except those in the last row. But this last row of spins gives rise to a factor $(2 \cosh L')^m$ which is positive and therefore can be neglected in the bulk limit. Now each decimated triangle contributes a factor F to the partition function Z. We finally obtain from (2) and (6) the expression

$$\kappa = [-2(\sinh 2K_1 \sinh 2K_2 + \sinh 2K \sinh 2L')/\sinh 2K_3]^{1/2} \tag{14}$$

where L' is to be determined from (8). This solution is valid along the trajectory (13), which can be rewritten as, using (7),

$$\sinh L = \frac{e^{-K_1-K_2} \sinh(2L'+K) - e^{K_1+K_2} \sinh K}{[2 \cosh 2(K_1+K_2) + 2 \cosh 2(K+L')]^{1/2}} \tag{15}$$

The solution is confined to the regions specified by (10) and by (11) and (12).

3. DISCUSSIONS

The solution (14) is analytic in K and K_i so there is no phase transition *along* the trajectory (15). It is instructive to examine the solution in some special cases.

(a) $J_3 = 0$. By taking the appropriate limits of (8) and (14), we find

$$\kappa = [2(\cosh 2K_1 \cosh 2K_2 + \cosh 2K \cosh 2L')]^{1/2} \tag{16}$$

where

$$\sinh 2L' = -\sinh 2K_1 \sinh 2K_2/\sinh 2K \tag{17}$$

The magnetic field L is obtained by substituting (17) into (15), and, as a consequence of (10), the solution (16) is valid for any $J \neq 0$.

(b) $J = 0$. This is the Ising model with pure two-site interactions. Now, the relations from which we determine L are the same as those given in Ref. 2,[5] so we are led to the same soluble trajectory as that of Ref. 2. But our procedure permits a direct determination of the free energy. We find

$$\kappa = [-2 \sinh 2K_1 \sinh 2K_2/\sinh 2K_3]^{1/2} \tag{18}$$

[5] This can be seen by comparing (4) and (13) with (2.8), (2.9), and (2.10) of Ref. 2.

and

$$\sinh^2 L = \frac{-4}{(e^{4K_1}-1)(e^{4K_2}-1)(e^{4K_3}-1)} \times [e^{4K_3}\cosh^2(K_1+K_2)-\cosh^2(K_1-K_2)] \times [e^{4K_3}\sinh^2(K_1+K_2)-\sinh^2(K_1-K_2)] \tag{19}$$

For a given Ising model with fixed J_1, J_2, and J_3, the expression (18) gives rise to three distinct solutions, obtained by permuting K_1, K_2, and K_3. These three solutions correspond to different values of the boundary field L', and the condition (3) is now used to single out the one for the model (1).

Direct evaluation using (8) leads to

$$\cosh 2L' = -\frac{e^{-2K_3}\sinh 2K_1 \sinh 2K_2}{\sinh 2K_3} - \cosh 2(K_1+K_2) \tag{20}$$

Hence (3) implies that we must have

$$J_1 J_2 J_3 < 0 \tag{21}$$

Therefore our result applies to the *antiferromagnetic* model (21). Using (20) and (21), the validity condition (3) can be simplified as

$$|e^{4K_3}-1| < \left|\frac{\cosh^2(K_1-K_2)}{\cosh^2(K_1+K_2)} - 1\right| \tag{22}$$

It can be verified that (22) can hold only when

$$|J_3| < |J_1|, |J_2| \tag{23}$$

and

$$T \leqslant T_D \tag{24}$$

where T_D is the disorder temperature of the antiferromagnetic Ising system.[9] In the region specified by (21) and (23) the disorder temperature is given by

$$e^{2K_3} = \cosh(K_1-K_2)/\cosh(K_1+K_2) \tag{25}$$

It can also be verified that the right-hand side of (19) is nonnegative in the region (21) and (24). Therefore (19) always leads to real values of L. Particularly, we find $L=0$ at $T=T_D$ and that the two other $T=T_D$ solutions, obtained by permuting K_1, K_2, and K_3 in (18) and (19), correspond to the boundary condition $e^{2L'} = -1$, and hence are not valid solutions. [Permutation of K_1, K_2, and K_3 in (19) leads to the same T_D when $L = 0$.] At $T = 0$ the trajectory (19) terminates at $H = \pm H_c$, where H_c is the maximum field for which a doubly degenerate (square-order) phase will occur at $T = 0$. $H_c = 2|J_1| + 2|J_2| - 4|J_3|$ if all $J_i < 0$; $H_c = 2|J_\alpha| - 2|J_3|$, if only one $J_\alpha < 0$.[10] Therefore the trajectory (19) does not appear to intersect the phase boundary.

(c) $J = \pm J_3$. In this case the solution is given by (14) and (15) with, from (8),

$$e^{\pm 2L'} = \frac{e^{-2K_3}\cosh 2(K_1 - K_2) - e^{2K_3}\cosh 2(K_1 + K_2)}{\sinh 4K_3} \tag{26}$$

Again, we find from (3) that the solution is valid for the model specified by (21) and (23), and is confined in the regions

$$T \leqslant T'_D \tag{27}$$

where T'_D is the temperature defined by

$$e^{4K_3} = \cosh 2(K_1 - K_2)/\cosh 2(K_1 + K_2) \tag{28}$$

4. SUMMARY

We have succeeded in computing the partition function per site for the Ising model (1). The solution, given by (14) and (15), is valid under the condition (3), or, equivalently, (10)–(12). For the nearest-neighbor model the condition specifies the antiferromagnetic model (21) and (23) at $T \leqslant T_D$. The condition (23) indicates that decimations must be used with care, and that it is valid only when carried out along one preferred lattice direction.

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